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Maximum Speed of Quantum Evolution

M. Andrecut[1] and M. K. Ali[1]

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In this paper, we discuss the question of the minimum time needed for any state of a given quantum system to evolve into a distinct (orthogonal) state. This problem is relevant to deriving physical limits in quantum computation and quantum information processing. Here, we consider both cases of nonadiabatic and adiabatic evolution and we derive the Hamiltonians corresponding to the minimum time evolution predicted by the Margolus–Levitin theorem.

KEY WORDS: quantum computation; adiabatic quantum evolution.

1. INTRODUCTION

In this paper, we discuss the question of the minimum time needed for any state of a given quantum system to evolve into a distinct (orthogonal) state. The problem of maximum speed of quantum dynamical evolution is relevant to deriving physical limits in quantum computation and quantum information processing. Recently, Margolus and Levitin (1997) have derived the minimum time required for the evolution from any state to an orthogonal state of the quantum system. However, they have considered only the nonadiabatic case and they have not given a Hamiltonian corresponding to the minimum evolution time. Here, we consider both cases of nonadiabatic and adiabatic evolution and we derive the Hamiltonians corresponding to the minimum time evolution predicted by the Margolus–Levitin theorem.

2. THE NONADIABATIC CASE

Consider a quantum system, which evolves according to the Schrödinger equation

$$i\frac{d}{dt}|\xi(t)\rangle = \hat{H}(t)|\xi(t)\rangle \tag{1}$$

[1] Department of Physics, University of Lethbridge, Lethbridge, AB, T1K 3M4, Canada; e-mail: mircea.andrecut@uleth.ca; ali@uleth.ca.

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where $\hat{H}$ is the Hamiltonian of the system (we let $\hbar = 1$). For the nonadiabatic case, we assume that $\hat{H}$ is time-independent. Also, we consider the system in the initial state $|\xi(0)\rangle = |\psi\rangle$, of mean energy $E = \langle\psi\hat{H}|\psi\rangle$. The Margolus–Levitin theorem (Margolus and Levitin, 1997) asserts that it takes at least a time $T_\perp \geq \pi/(2E)$ for the system to evolve from, $\xi(0)\rangle = |\psi\rangle$ to an orthogonal state, $\xi(T_\perp)\rangle = |\varphi\rangle$. This result complements the time–energy uncertainty relation, which requires $T_\perp \geq \pi/(2\Delta E)$, where $\Delta E = \sqrt{\langle\psi(\hat{H} - E)^2|\psi\rangle}$ is the energy spread of the state. Combining the two above inequalities one obtains the minimum time required for the evolution to an orthogonal state (Margolus and Levitin, 1998):

$$T_\perp = T(E, \Delta E) = \max\left(\frac{\pi}{2E}, \frac{\pi}{2\Delta E}\right) \tag{2}$$

However, the Margolus–Levitin theorem does not provide a Hamiltonian corresponding to the minimum evolution time. Farhi and Gutmann (1996) showed that the following Hamiltonian

$$\hat{H} = E(|\psi\rangle\langle\psi| + |\varphi\rangle\langle\varphi|) \tag{3}$$

evolves the state $|\psi\rangle$ into the state $|\varphi\rangle$ after a time

$$T_{FG}(E, \alpha) = \frac{\pi}{2E\alpha} \tag{4}$$

where $\alpha = \langle\psi|\varphi\rangle$. This implies that

$$\lim_{\alpha\to 0} T_{FG}(E, \alpha) = \infty \tag{5}$$

Thus, if the states $|\psi\rangle$ and $|\varphi\rangle$ are orthogonal, the evolution fails.

Now, let us consider the Hamiltonian

$$\hat{H} = E\left(e^{-i\theta}|\psi\rangle\langle\varphi| + e^{i\theta}|\varphi\rangle\langle\upsilon|\right) \tag{6}$$

where $E > 0, \theta$ is a constant phase, and $|\psi\rangle$ and $|\varepsilon\rangle$ are orthogonal: $\langle\psi|\varphi\rangle = 0$. After a time t, the state of the quantum system, evolving according to this Hamiltonian and initially in the state $|\xi(0)\rangle = \psi\rangle$, is given by:

$$|\xi(t)\rangle = \exp(-i\hat{H}t)|\psi\rangle \tag{7}$$

It is well known, that for any operator $\hat{A}$, with $\hat{A}^2 = \hat{I}$ (where $\hat{I}$ is the identity operator) we have

$$\exp(it\hat{A}) = \cos(t)\hat{I} + i\sin(t)\hat{A} \tag{8}$$

Now, taking into account that $\hat{H}^2 = E^2\hat{I}$, a simple calculation gives

$$|\xi(t)\rangle = \cos(Et)|\psi\rangle - ie^{i\theta}\sin(Et)|\varphi\rangle \tag{9}$$

Taking $\theta = \pi/2$, the minimum evolution time from $|\psi\rangle$ to $|\varphi\rangle$ is

$$T(E) = \frac{\pi}{2E} \tag{10}$$

which is exactly the time predicted by the Margolus–Levitin theorem.

3. THE ADIABATIC CASE

Consider a quantum system in a state $|\xi(t)\rangle$, which evolves according to the Schrödinger equation (1). If the Hamiltonian is time-independent and the system is initially in its ground state, then it will remain in this state. More specifically, if $|E_0; t\rangle$ and $|E_1; t\rangle$ are the ground and first excited states of the Hamiltonian $\hat{H}(t)$, with energies E_0 and E_1, we define the minimum gap between these eigenvalues

$$\omega_{\min} = \min_{0 \le t \le T} [E_1(t) - E_0(t)] \tag{11}$$

and the maximum value of the matrix element of $d\hat{H}(t)/dt$ between the eigenstates as

$$\Omega_{\max} = \max_{0 \le t \le T} \left| \left\langle \frac{d\hat{H}}{dt} \right\rangle_{1,0} \right| = \max_{0 \le t \le T} \left| \langle E_{1;t} | \frac{d\hat{H}}{dt} | E_0; t \rangle \right| \tag{12}$$

The adiabatic theorem states that if we prepare the system at time $t = 0$ in its ground state $|E_0; t\rangle$ and let it evolve under the Hamiltonian $\hat{H}(t)$ for a time T, then

$$|\langle E_0; T | \xi(T) \rangle|^2 \ge 1 - \varepsilon^2 \tag{13}$$

provided that

$$\Omega_{\max} \omega_{\min}^{-2} \le \varepsilon \tag{14}$$

where $0 < \varepsilon \ll 1$ (Messiah, 1976; Bransden and Joachain, 2000).

This result can be used to design a new type of quantum algorithm based on a time-dependent Hamiltonian (Farhi *et al*., 2000). Assume we can build a Hamiltonian for which we know that the ground state encodes the solution of a problem. Then, it suffices to prepare the system in the ground state of another Hamiltonian, easy to build, and change progressively this Hamiltonian into the other one in order to get, after measurement, the sought solution with large probability. The adiabatic theorem imposes the minimum time it takes for this switching to be adiabatic.

The adiabatic method has been studied only for the linear interpolation case, where the "straight line" interpolation from initial ($\hat{H}_0$) to final ($\hat{H}_1$) Hamiltonian is taken (Farhi *et al*., 2000; Roland and Cerf, 2001; van Dam, Mosca and Vazirani, 2001):

$$\hat{H}(s) = (1 - s)\hat{H}_0 + s\hat{H}_1 \tag{15}$$

where $s = t/T$ is the rescaled time and T is the total time (or *delay schedule*). The initial state of the system $|\xi(0)\rangle = |\psi\rangle$ is the ground state, with energy $E_0 = -E(E > 0)$, of the initial Hamiltonian

$$\hat{H}_0 = -E|\psi\rangle\langle\psi| \tag{16}$$

We would like to evolve this state, using the linear interpolating Hamiltonian, to the final state $|\xi(1)\rangle = |\varphi\rangle$, which is the ground state, with energy $E_0 = -E$, of the final Hamiltonian

$$\hat{H}_1(\theta) = -E|\varphi\rangle\langle\varphi| \tag{17}$$

We assume that $|\psi\rangle$ and $|\varphi\rangle$ are not orthogonal: $\langle\psi|\varphi\rangle = \cos(\theta)$. Therefore we can write

$$|\varphi\rangle = \cos(\theta)|\psi\rangle + \sin(\theta)|\sigma\rangle \tag{18}$$

where $\langle\psi|\sigma\rangle = 0$. In the two-dimensional space spanned by $|\sigma\rangle$ and $|\psi\rangle$ the linear interpolating Hamiltonian reads

$$\hat{H}(s,\theta) = -E\begin{bmatrix} 1 - s\sin^2(\theta) & \frac{1}{2}s\sin(2\theta) \\ \frac{1}{2}s\sin(2\theta) & s\sin^2(\theta) \end{bmatrix} \tag{19}$$

The ground state and the first excited state of the above Hamiltonian are

$$E_{0,1}(s,\theta) = \frac{E}{2}\left[1 \mp \sqrt{1 - 4s(1-s)\sin^2(\theta)}\right] \tag{20}$$

Thus the energy gap is given by

$$\omega(s,\theta) = E\sqrt{1 - 4s(1-s)\sin^2(\theta)} \tag{21}$$

Also, it is easy to show that the matrix element of $d\hat{H}(t)/dt$ between the eigenstates is given by

$$\Omega(t,\theta) = \left|\frac{ds}{dt}\right|\Omega(s,\theta) = \frac{E}{T}\left|\frac{\sin(2\theta)}{2\omega(s,\theta)}\right| \tag{22}$$

The extreme values are obtained for $s = 1/2$:

$$\omega_{\min}(\theta) = E\cos(\theta) \tag{23}$$

$$\Omega_{\max}(\theta) = \frac{E}{T}|\sin(\theta)| \tag{24}$$

By substituting (23) and (24) into the adiabaticity condition (14), we obtain the minimum evolution time for the linear interpolation case (Farhi *et al.*, 2000):

$$T_0(E,\theta) = \frac{1}{E\varepsilon}\frac{|\sin(\theta)|}{\cos^2(\theta)} \tag{25}$$

A better result can be obtained by assuming that $s = s(t)$ is a smooth function satisfying the boundary conditions: $s(0) = 0$, $s(T) = 1$ (Rolard and Cerf, 2001; van Dam, Mosca, and Vazirani, 2002). In this case, the local version of the adiabatic evolution condition reads

$$\left|\frac{ds}{dt}\right| \leq \varepsilon \frac{\omega^2(s,\theta)}{\Omega(s,\theta)} \tag{26}$$

and the minimum running time can be obtained by integration (Roland and Cerf, 2001; van Dam, Mosca and Vazirani, 2002):

$$T_1(E_\theta) = \int_0^1 ds \frac{\Omega(s,\theta)}{\omega^2(s,\theta)} = \frac{1}{E\varepsilon}\tan(\theta) \tag{27}$$

However, one can see that when the initial and final states are orthogonal the minimum running time becomes infinite:

$$\lim_{\theta\to\pi/2} T_{0,1}(E,\theta) = \infty \tag{28}$$

In fact, in the orthogonal case ($\theta = \pi/2$) the minimum gap is $\omega_{\min}(\pi/2) = 0$ and the adiabaticity conditions (14) and (26) are not satisfied. Thus, the adiabatic evolution (based on the above described linear interpolation method) fails when the initial and final states are orthogonal.

In what follows we will show that the adiabatic evolution can be done in finite time by using a simple nonlinear interpolation method. There is no reason not to consider nonlinear interpolation. The adiabatic algorithm will work taking any path $\hat{H}(s)$, as long as the adiabaticity condition is satisfied.

We assume that $|\psi\rangle$ and $|\varphi\rangle$ are orthogonal: $\langle\psi|\varphi\rangle = 0$. Now, let us consider the following Hamiltonian:

$$\hat{H}(s) = -E|\xi(s)\rangle\langle\xi(s)| \tag{29}$$

where

$$|\xi(s)\rangle = \cos\left(\frac{\pi}{2}s\right)|\psi\rangle + \sin\left(\frac{\pi}{2}s\right)|\varphi\rangle \tag{30}$$

is the ground state, with eigenvalue $E_0 = -E(E > 0)$, and $s = t/T \in [0, 1]$.

In the $|\psi\rangle$, $|\varphi\rangle$ basis, the adiabatic Hamiltonian reads

$$\begin{aligned}\hat{H}(s) = &-E\sin^2\left(\frac{\pi}{2}s\right)|\psi\rangle\langle\psi| - E\cos^2\left(\frac{\pi}{2}s\right)|\varphi\rangle\langle\varphi| \\ &-E\sin(\pi s)(|\psi\rangle\langle\varphi| + |\varphi\rangle\langle\psi|)\end{aligned} \tag{31}$$

Thus, $|\xi(s)\rangle$ performs an interpolation from the initial wave function $|\psi\rangle = |\xi(0)\rangle$ to final wave function $|\varphi\rangle = |\xi(1)\rangle$, and $\hat{H}(s)$ performs an interpolation from the initial Hamiltonian $\hat{H}_0 = \hat{H}(0) = -E|\psi\rangle\langle\psi|$ to final Hamiltonian $\hat{H}_1 = \hat{H}(1) = -E|\varphi\rangle\langle\varphi|$.

The first excited state is $E_1 = 0$. Thus, the energy gap is constant: $\omega_{\min} = E$. Also, taking into account that

$$\frac{d\hat{H}}{dt} = \frac{ds}{dt}\frac{d\hat{H}}{ds} = \frac{1}{T}\frac{d\hat{H}}{ds} \tag{32}$$

the matrix element can be calculated analytically and it is given by

$$\Omega_{\max} = \frac{\pi E}{2T} \tag{33}$$

Thus, the minimum evolution time from $|\psi\rangle$ to $|\varphi\rangle$, in the adiabatic case, is

$$T(E) = \frac{\pi}{2E\varepsilon} \tag{34}$$

where E is the minimum gap between the first excited state and the ground state of the system.

4. CONCLUSIONS

In this paper, we have discussed the question of the minimum time needed for any state of a given quantum system to evolve into a distinct (orthogonal) state. First, we have considered the case of nonadiabatic evolution and we have derived the Hamiltonian corresponding to the minimum time evolution predicted by the Margolus–Levitin theorem. In the adiabatic case we have proposed a simple nonlinear interpolation method which gives an analogue result to the Margolus–Levitin theorem.

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